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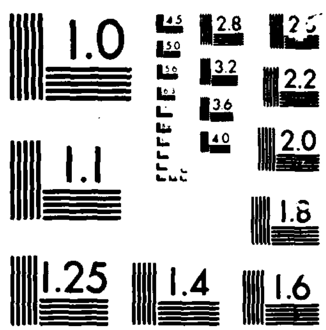

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THE EVALUATION OF INNER PRODUCTS OF MULTIVARIATE SIMPLEX SPLINES

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Thomas A. Grandine[1,2]

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ABSTRACT

This paper gives a general method for the stable evaluation of inner products of multivariate simplex splines. The method is based on a recurrence relation for these inner products. The base cases for this recurrence relation are handled by triangulating certain simploids into simplices.

AMS(MOS) Subject Classification: 41A15, 65D07

Key Words: B-spline, simplex spline, simploid spline, multivariate, recurrence relation, linear programming, simplex method

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SIGNIFICANCE AND EXPLANATION

For many types of projection methods (Rayleigh-Ritz-Galerkin, for example), it is necessary to compute inner products of the basis elements in order to be able to implement the scheme. This paper discusses a method for evaluating these inner products in the case where the basis elements are simplex splines.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE EVALUATION OF INNER PRODUCTS OF MULTIVARIATE SIMPLEX SPLINES

Thomas A. Grandine[1,2]

In many practical problems, it is necessary to be able to compute the value of the inner product of any two simplex splines. This is typically the case for many types of projection methods, including Rayleigh-Ritz-Galerkin and least squares. While various types of quadrature methods can be used for this purpose, Dahmen and Micchelli [DM81] report that far better results are obtained if exact values of these inner products are used. Developing an algorithm for evaluating these inner products is the goal of this paper.

The multivariate polyhedral B-spline M_B is defined as a **distribution** by

$$\int_{\mathbf{R}^m} M_B f := \int_B f \circ P, \tag{1}$$

where f is an arbitrary m-variate function, B is a polyhedral body in $\mathbf{R}^n$, $n \geq m$, and $P : \mathbf{R}^n \to \mathbf{R}^m$ is a linear map. Typically, P is chosen to be the **canonical projector**, i.e. $P : x \mapsto x(i)_{i=1}^m$. This is a generalization of an earlier definition due to Micchelli in which B is chosen specifically to be a simplex. In particular, if x_0, x_1, ..., x_n are points in $\mathbf{R}^n$, and if $[A]$ denotes the convex hull of A, then the multivariate simplex spline $M(\cdot|x_0, ..., x_n)$ is defined to be [M80]:

$$\int_{\mathbf{R}^m} M(\cdot|x_0, ..., x_n) f := \frac{1}{\mathrm{vol}_n [x_0, ..., x_n]} \int_{[x_0, ..., x_n]} f \circ P. \tag{2}$$

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This definition, which is a convenient reformulation of an earlier definition of de Boor [B76], is a special case of definition (1) except for the difference in normalization.

The purpose of this paper is to describe a method for evaluating inner products of simplex splines (with the normalization given in (2)). The inner product referred to here is the one given by the following integral:

$$\langle f, g \rangle := \int_{\mathbf{R}^m} fg.$$

An identity can be used to evaluate the integral in the case where f and g happen to be two polyhedral splines, say M_B and M_C, defined as in (1) by

$$\int_{\mathbf{R}^m} M_B f := \int_B f \circ P$$
$$\int_{\mathbf{R}^m} M_C f := \int_C f \circ Q,$$

for two ordinary linear maps P and Q into $\mathbf{R}^m$. From [DM82-2], it is known that the convolution of two polyhedral splines satisfies

$$M_{B\times C}(x) = \int_{\mathbf{R}^m} M_B(y-x) M_C(y) dy, \tag{3}$$

where $B \times C$ is the polyhedral set which is the usual **Cartesian product** of the polyhedral sets B and C, i.e. $B \times C := \{(r,s) | r \in B, s \in C\}$. Setting $x = 0$ in this equation results in the identity

$$\int_{\mathbf{R}^m} M_B(x) M_C(x) dx = M_{B\times C}(0), \tag{4}$$

where

$$\int_{\mathbf{R}^m} M_{B\times C}(x) f(x) dx = \int_B \int_C f(Qy - Px) dy dx. \tag{5}$$

If B and C are simplices, then this relation says that the value of the inner product of two simplex splines is given by the value of a certain simploid spline at the origin. A **simploid** is defined as the Cartesian product of two simplices. Thus, if an accurate way can be found to evaluate this simploid spline, then an accurate way has been found to compute the value of the desired inner product.

Dahmen and Micchelli [DM81-2] have proposed the use of the recurrence relation for polyhedral splines for this purpose. Specialized to the case of the simploid, and then rewritten in terms of inner products, this recurrence relation takes on the following form [DM81-2]:

Theorem 1: If $B = [x_0, ..., x_n]$, $C = [y_0, ..., y_p]$, and $\sum_{i=0}^{n} \alpha_i x_i = \sum_{j=0}^{p} \beta_j y_j$, with $\sum_{i=0}^{n} \alpha_i = \sum_{j=0}^{p} \beta_j = 1$, then

$$\begin{aligned}\int_{\mathbf{R}^m} M(x|x_0,...,x_n)M(x|y_0,...,y_p)dx = \frac{1}{n+p-m}\Big(\\ n\sum_{i=0}^{n}\alpha_i \int_{\mathbf{R}^m} M(x|x_0,...,x_{i-1},x_{i+1},...,x_n)M(x|y_0,...,y_p)dx+ \\ p\sum_{j=0}^{p}\beta_j \int_{\mathbf{R}^m} M(x|x_0,...,x_n)M(x|y_0,...,y_{j-1},y_{j+1},...,y_p)dx\Big).\end{aligned} \tag{6}$$

This recurrence relation can be implemented using the linear programming approach of [G84]. This technique requires, at the first step, a solution to the following problem: Find α_i and β_j so that

$$\begin{aligned}\sum_{i=0}^{n}\alpha_i x_i - \sum_{j=0}^{p}\beta_j y_j &= 0\\ \sum_{i=0}^{n}\alpha_i &= 1\\ \sum_{j=0}^{p}\beta_j &= 1\\ \alpha_i \geq 0,& \quad i = 0,...,n\\ \beta_j \geq 0,& \quad j = 0,...,p.\end{aligned} \tag{7}$$

As described in [G84], problems of this sort can be solved via the dual simplex method for linear programming problems. This method always results in solutions for which all but $m + 2$ of the α_i and β_j are zero. This is because the problem (7) has only $m + 2$ equality constraints in it, and solutions to linear programming problems can always be made to satisfy a complementarity condition (see [Da63] and [Ma78]). This number is a drastic improvement over the $2m + 2$ given by Dahmen and Micchelli in [DM81].

In any case, the implementation of the recurrence relation (6) proceeds just as in [G84]. The snag appears when one or the other (or both) of the simplex splines appearing on the left-hand-side of (6) are piecewise constant functions, which happens when a simplex spline has exactly $m + 1$ **knots** present. In this case, the recurrence relation cannot be applied, for then some of the functions appearing on the right-hand-side of the recurrence relation will not be well-defined. Dahmen and Micchelli conquer this obstacle by expressing each of the simplex splines in the integral on the left-hand-side of (6) as a linear combination of cone splines. Then, since the convolution of two cone splines is again a cone spline, they are able to write this inner product as a linear combination of cone splines (see [D79], [D80], and [DM81-2]), each of which may be evaluated using the recurrence relation for cone splines.

There is a way, however, to avoid going to the trouble of implementing the recurrence relation for cone splines. In fact, the cone splines can be dispensed with altogether by noting, as was done already in [H82] and [DM82], that any simploid can easily be triangulated merely by arranging its vertices in a rectangular grid and tracing the various paths through this grid. This makes it possible to express any simploid spline as a linear combination of simplex splines, each of which may be evaluated using the method described in [G84].

In what follows, let $\Sigma_1 := [x_0, ..., x_n]$ and $\Sigma_2 := [y_0, ..., y_p]$. With $t := \binom{n+p}{n}$, let $\sigma_1, ..., \sigma_t$ be a collection of simplices such that $\{\sigma_i\}$ is the triangulation of $\Sigma_1 \times \Sigma_2$ that is produced via the following construction [H82]: Consider the following grid, representing the set $\Sigma_1 \times \Sigma_2$,

$$\begin{array}{cccc} (x_0, y_p) & (x_1, y_p) & \dots & (x_n, y_p) \\ \vdots & \vdots & \ddots & \vdots \\ (x_0, y_1) & (x_1, y_1) & \dots & (x_n, y_1) \\ (x_0, y_0) & (x_1, y_0) & \dots & (x_n, y_0) \end{array}.$$

Take all the **paths** through this grid given by $\sigma = \{s_0, ..., s_{n+p}\}$, where $s_0 = (x_0, y_0)$, $s_{n+p} = (x_n, y_p)$, and if $s_i = (x_j, y_\ell)$, then s_{i-1} must be either (x_{j-1}, y_ℓ) or $(x_j, y_{\ell-1})$. There are $\binom{n}{m}$ different paths through the grid. It can be shown that the set of all paths through the grid is a triangulation of the simploid (Lemma 3 in [H82]).

This construction ensures the truth of

Lemma 1:

$$\text{vol}\sigma_i = \frac{\text{vol}\Sigma_1 \times \Sigma_2}{t}.$$

Proof: Let P_j be the canonical projector from $\Sigma_1 \times \Sigma_2$ into Σ_j. Then

$$\begin{aligned} \frac{\text{vol}\Sigma_1 \times \Sigma_2}{t} &= \frac{\text{vol}\Sigma_1 \cdot \text{vol}\Sigma_2}{t} \\ &= \frac{n!\text{vol}P_1\sigma_i \cdot p!\text{vol}P_2\sigma_i}{(n+p)!} \\ &= \text{vol}\sigma_i. \end{aligned}$$

With this lemma, it is possible to prove

Theorem 2:

$$\int_{\mathbf{R}^m} M(x|x_0, ..., x_n) M(x|y_0, ..., y_p) dx = (1/t) \sum_{i=1}^{t} \frac{M_{\sigma_i}(0)}{\text{vol}\sigma_i}. \tag{8}$$

Proof: By definition,

$$\begin{aligned}
\int_{\mathbf{R}^m} M_{\Sigma_1\times\Sigma_2} f &= \int_{\Sigma_1\times\Sigma_2} f\circ P \\
&= \sum_{i=1}^{t}\int_{\sigma_i} f\circ P \\
&= \int_{\mathbf{R}^m}\left(\sum_{i=1}^{t} M_{\sigma_i}\right) f,
\end{aligned}$$

so that $M_{\Sigma_1\times\Sigma_2} = \sum_{i=1}^{t} M_{\sigma_i}$. Then

$$\begin{aligned}
\int_{\mathbf{R}^m} M(x|x_0,\ldots,x_n)M(x|y_0,\ldots,y_p)dx &= \int_{\mathbf{R}^m}\frac{M_{\Sigma_1}M_{\Sigma_2}}{\mathrm{vol}\Sigma_1\mathrm{vol}\Sigma_2}dx \\
&= \frac{1}{\mathrm{vol}\Sigma_1\times\Sigma_2}M_{\Sigma_1\times\Sigma_2}(0) \\
&= \frac{1}{\mathrm{vol}\Sigma_1\times\Sigma_2}\sum_{i=1}^{t}M_{\sigma_i}(0) \\
&= \frac{1}{\mathrm{vol}\Sigma_1\times\Sigma_2}\sum_{i=1}^{t}\mathrm{vol}\sigma_i\frac{M_{\sigma_i}(0)}{\mathrm{vol}\sigma_i} \\
&= \frac{1}{t}\sum_{i=1}^{t}\frac{M_{\sigma_i}(0)}{\mathrm{vol}\sigma_i}.
\end{aligned}$$

As it turns out, the terms on the right-hand-side of this formula are all normalized in the standard way for simplex splines. Hence, the simplex splines appearing on the right-hand-side of (8) are those obtained by considering all possible paths through the grid:

$$\begin{array}{cccc}
x_0 - y_p & x_1 - y_p & \cdots & x_n - y_p \\
\vdots & \vdots & \ddots & \vdots \\
x_0 - y_1 & x_1 - y_1 & \cdots & x_n - y_1 \\
x_0 - y_0 & x_1 - y_0 & \cdots & x_n - y_0
\end{array}$$

Formula (8) provides an easy way of evaluating inner products when the recurrence relation (6) cannot be used because one or more of the simplex splines in the integral have only $m+1$ knots. It also provides an easy way to dispense with (6) altogether, since (8) is true independent of the number of knots appearing in the simplex splines in the integral. As elegant and clean as this approach is, however, it is much less efficient than using recurrence relation (6) whenever possible, and then using relation (8) the rest of the time.

This is best seen by considering the following calculation. Suppose that the value of an inner product of a simplex spline with $n+1$ knots and a simplex spline of $p+1$ knots is desired. To obtain this value using formula (8) will require the evaluation of $\binom{n+p}{n}$ simplex splines of degree $n+p-m$. Each of these simplex splines requires $m+1$ times as much work to evaluate as does a simplex spline of degree $n+p-m-1$. Thus, the total amount of work required is roughly equivalent to that of evaluating $(m+1)\binom{n+p}{n}$ simplex splines of degree $n+p-m-1$. Now consider what happens when one application of the recurrence relation (6) is applied, followed by this triangulation technique. The recurrence relation will produce a sum of inner products, say δ_1 inner products of $n+1$ knot and p knot simplex splines, and δ_2 inner products of n knot and $p+1$ knot simplex splines. Here $\delta_1 + \delta_2 = m+2$, as remarked earlier. After triangulating each of these simploids, the total number of degree $n+p-m-1$ simplex splines which need to be evaluated is $\delta_1\binom{n+p-1}{n} + \delta_2\binom{n+p-1}{p}$. Since $\delta_1 \leq m+1$ and $\delta_2 \leq m+1$, with equality in at most one of them, it follows that $\delta_1 p + \delta_2 n < (m+1)(n+p)$. Multiplying this inequality by $(n+p-1)!/(n!p!)$ yields

$$\delta_1\binom{n+p-1}{n} + \delta_2\binom{n+p-1}{p} < (m+1)\binom{n+p}{n}. \tag{9}$$

Thus, it is always more efficient to employ recurrence relation (6) whenever that is possible.

Example: Suppose that $n = p = 5$ and $m = 2$ above. Then using the recurrence relation (6) once requires evaluation of 504 simplex splines of degree 7, while not using it at all requires evaluation of 756 simplex splines of degree 7, half again as much work. Of course, it would make sense to employ (6) as often as possible to compute inner products. The point of this example is that even one application can make a tremendous difference.

References

[B76] de Boor, Carl. "Splines as linear combinations of B-splines," in *Approximation Theory II*, G. G. Lorentz. C. K. Chui, and L. L. Schumaker, eds., Academic Press, pp. 1-47.

[B82] de Boor, Carl. "Topics in Multivariate Approximation Theory," in *Topics in Numerical Analysis*, P. Turner, ed. *Springer Lecture Notes in Mathematics* **965**, 1982, pp. 39-78.

[BH81] de Boor, Carl, and Höllig, Klaus. "Recurrence Relations for Multivariate B-splines," *Proc. Amer. Math. Soc.* **85**, 1982, pp. 397–400.

[D79] Dahmen, W. "Multivariate B-splines – Recurrence Relations and Linear Combinations of Truncated Powers," in *Multivariate Approximation Theory*, W. Schempp and K. Zeller, eds., Basel: Birkhäuser, pp. 64–82.

[DM81] Dahmen, W., and Micchelli, C. A. "Numerical Algorithms for Least Squares Approximation by Multivariate B-splines," in *Numerical Methods of Approximation Theory*, Vol. 6, ed. by Collatz, Meinardus, and Werner, Basel: Birkhäuser, 1981.

[DM81-2] Dahmen, W., and Micchelli, C. A. "Computation of Inner Products of Multivariate B-splines," *Numer. Funct. Anal. and Optimiz.* **3(3)**, pp. 367–375.

[DM82] Dahmen, W., and Micchelli, C. A. "On the Linear Independence of Multivariate B-splines I., Triangulation of Simploids," *SIAM Journal of Numerical Analysis* **19**, pp. 993-1012.

[DM82] Dahmen, W., and Micchelli, C. A. "Some Remarks on Multivariate B-splines," in *Multivariate Approximation Theory II*, W. Schemp, K. Zeller, eds. Basel: Birkhäuser, 1982. pp. 81–87.

[Da63] Dantzig, G. B. *Linear Programming and Extensions.* Princeton University Press, 1963.

[G84] Grandine, Thomas. "The Stable Evaluation of Multivariate B-splines," Mathematics Research Center TSR #2744, 1984.

[H82] Höllig, Klaus. "Multivariate Splines," *SIAM Journal of Numerical Analysis* **19**, pp. 1013–1031.

[M78] Mangasarian, Olvi. "Linear Programming Lecture Notes." Manuscript, 1978.

[M80] Micchelli, C. A. "A Constructive Approach to Kergin Interpolation in $\mathbf{R}^k$: Multivariate B-splines and Lagrange Interpolation," *Rocky Mountains J. Math.* **10**, pp. 485–497.

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This paper gives a general method for the stable evaluation of inner products of multivariate simplex splines. The method is based on a recurrence relation involving inner products. The base cases for this recurrence relation are handled by triangulating certain simploids into simplices.

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